

# The general Liénard polynomial system<sup>★</sup>

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## Abstract

In this paper, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve first the problem on the maximum number of limit cycles surrounding a unique singular point for an arbitrary polynomial system. Then, by means of the same bifurcationally geometric approach, we solve the limit cycle problem for a general Liénard polynomial system with an arbitrary (but finite) number of singular points. This is related to the solution of Hilbert's sixteenth problem on the maximum number and relative position of limit cycles for planar polynomial dynamical systems.

*Keywords:* Hilbert's sixteenth problem; planar polynomial dynamical system; general Liénard polynomial system; field rotation parameter; bifurcation; singular point; limit cycle

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## 1 Introduction

We will consider Liénard equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \quad (1.1)$$

In the phase plane, the representation of equation (1.1) is given by the dynamical system

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \quad (1.2)$$

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<sup>★</sup> This work was supported by the Netherlands Organization for Scientific Research (NWO). The author is also very grateful to the Max Planck Institute for Mathematics (Bonn) and the Johann Bernoulli Institute for Mathematics and Computer Science (Groningen) for hospitality and support during his stay in 2011–2012.

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There are many examples in the natural sciences and technology in which this and related systems are applied [1]–[13]. Such systems are often used to model either mechanical or electrical, or biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. They can be used, e. g., in certain mechanical systems, where  $f(x)$  represents a coefficient of the damping force and  $g(x)$  represents the restoring force or stiffness, when modeling wind rock phenomena and surge in jet engines [2], [10]. Such systems can be also used to model resistor-inductor-capacitor circuits with non-linear circuit elements. Recently, e. g., the Liénard system (1.2) has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator [12]. There are also some examples of using Liénard type systems in ecology and epidemiology [9].

In this paper, we suppose that system (1.2), where  $f(x)$  and  $g(x)$  are arbitrary polynomials of  $x$ , has an anti-saddle (a node or a focus, or a center) at the origin and write it in the form

$$\dot{x} = y, \quad \dot{y} = -x(1 + \beta_1 x + \dots + \beta_{2l} x^{2l}) + y(\alpha_0 + \alpha_1 x + \dots + \alpha_{2k} x^{2k}). \quad (1.3)$$

Note that for  $g(x) \equiv x$ , by the change of variables  $X = x$  and  $Y = y + F(x)$ , where  $F(x) = \int_0^x f(s) ds$ , (1.3) is reduced to an equivalent system

$$\dot{X} = Y - F(X), \quad \dot{Y} = -X \quad (1.4)$$

which can be written in the form

$$\dot{x} = y, \quad \dot{y} = -x + F(y) \quad (1.5)$$

or

$$\dot{x} = y, \quad \dot{y} = -x + \gamma_1 y + \gamma_2 y^2 + \gamma_3 y^3 + \dots + \gamma_{2k} y^{2k} + \gamma_{2k+1} y^{2k+1}. \quad (1.6)$$

Therefore, we can conclude that our previous results [14], [15] agree with the conjecture of [7] on the maximum number of limit cycles for the classical Liénard polynomial system (1.6). In [14], [15], we have presented a solution of Smale's thirteenth problem [13] proving that the Liénard system (1.6) with a polynomial of degree  $2k + 1$  can have at most  $k$  limit cycles. In [16]–[21], we have also presented a solution of Hilbert's sixteenth problem in the quadratic case of polynomial systems proving that for quadratic systems four is really the maximum number of limit cycles and (3:1) is their only possible distribution. We have established some preliminary results on generalizing our ideas and methods to special cubic, quartic and other polynomial dynamical systems as well. In [22], e. g., we have constructed a canonical cubic dynamical system of Kukles type and have carried out the global qualitative analysis of its special case corresponding to a generalized Liénard equation. In particular, it has

been shown that the foci of such a Liénard system can be at most of second order and that such system can have at most three limit cycles in the whole phase plane. Moreover, unlike all previous works on the Kukles-type systems, global bifurcations of limit and separatrix cycles using arbitrary (including as large as possible) field rotation parameters of the canonical system have been studied. As a result, a classification of all possible types of separatrix cycles for the generalized Liénard system has been obtained and all possible distributions of its limit cycles have been found. In [23], [24], we have completed the global qualitative analysis of a planar Liénard-type dynamical system with a piecewise linear function containing an arbitrary number of dropping sections and approximating an arbitrary polynomial function. In [25]–[29], we have also completed the global qualitative analysis of a quartic dynamical system which models the dynamics of the populations of predators and their prey in a given ecological system. In [30], we have studied of multiple limit cycle bifurcations in the well-known FitzHgh–Nagumo neuronal model.

We use the obtained results and develop our methods for studying limit cycle bifurcations of polynomial dynamical systems in this paper as well. In Section 2, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve first the problem on the maximum number of limit cycles surrounding a unique singular point for an arbitrary polynomial system. Then, in Section 3, by means of the same bifurcationally geometric approach, we solve the limit cycle problem for the general Liénard polynomial system (1.3) with an arbitrary (but finite) number of singular points proving that system (1.3) can have at most  $k + l$  limit cycles,  $k$  surrounding the origin and  $l$  surrounding one by one the other singularities of (1.3). This is related to the solution of Hilbert’s sixteenth problem on the maximum number and relative position of limit cycles for planar polynomial dynamical systems.

## 2 Limit cycles surrounding a unique singular point

Consider first an arbitrary dynamical system

$$\dot{x} = P_n(x, y, \mu_1, \dots, \mu_k), \quad \dot{y} = Q_n(x, y, \mu_1, \dots, \mu_k), \quad (2.1)$$

where  $P_n$  and  $Q_n$  are polynomials in the real variables  $x, y$  and not greater than  $n$ -th degree containing  $k$  field rotation parameters,  $\mu_1, \dots, \mu_k$ , and having an anti-saddle at the origin. We prove the following theorem.

**Theorem 2.1.** *The polynomial system (2.1) containing  $k$  field rotation parameters and having a singular point of center type at the origin for the zero*

values of these parameters can have at most  $k - 1$  limit cycles surrounding the origin.

**Proof.** Let all the parameters of (2.1) vanish and suppose that the obtained system

$$\dot{x} = P_n(x, y, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, 0, \dots, 0) \quad (2.2)$$

has a singular point of center type at the origin.

Input successively the field rotation parameters,  $\mu_1, \dots, \mu_k$ , into this system (see [14], [15]). Suppose, e.g., that  $\mu_1 > 0$  and that the vector field of the system

$$\dot{x} = P_n(x, y, \mu_1, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, 0, \dots, 0) \quad (2.3)$$

is rotated counterclockwise turning the origin into a stable focus under increasing the parameter  $\mu_1$  [1], [14], [15].

Fix  $\mu_1$  and input the parameter  $\mu_2$  into (2.3) changing it so that the field of the system

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, 0, \dots, 0) \quad (2.4)$$

would be rotated in the opposite direction (clockwise). Suppose this occurs for  $\mu_2 < 0$ . Then, for some value of this parameter, a limit cycle will appear in system (2.4). There are three logical possibilities for such a bifurcation: 1) the limit cycle appears from the focus at the origin; 2) it can also appear from some separatrix cycle surrounding the origin; 3) the limit cycle appears from a so-called “trajectory concentration”. In the last case, the limit cycle is semi-stable and, under further decreasing  $\mu_2$ , it splits into two limit cycles (stable and unstable) one of which then disappears at (or tends to) the origin and the other disappears on (or tends to) some separatrix cycle surrounding this point. But since the stability character of both a singular point and a separatrix cycle is quite easily controlled [18], this logical possibility can be excluded. Let us choose one of the two other possibilities: e.g., the first one, the so-called Andronov–Hopf bifurcation. Suppose that, for some value of  $\mu_2$ , the focus at the origin becomes non-rough, changes the character of its stability and generates a stable limit cycle,  $\Gamma_1$ .

Under further decreasing  $\mu_2$ , three new logical possibilities can arise: 1) the limit cycle  $\Gamma_1$  disappears on some separatrix cycle surrounding the origin; 2) a separatrix cycle can be formed earlier than  $\Gamma_1$  disappears on it and then it generates one more (unstable) limit cycle,  $\Gamma_2$ , which joins with  $\Gamma_1$  forming a semi-stable limit cycle,  $\Gamma_{12}$ , disappearing in a “trajectory concentration” under further decreasing  $\mu_2$ ; 3) in the domain  $D_1$  outside the cycle  $\Gamma_1$  or in the domain  $D_2$  inside  $\Gamma_1$ , a semi-stable limit cycle appears from a “trajectory concentration” and then splits into two limit cycles (logically, the appearance of such semi-stable limit cycles can be repeated).

Let us consider the third case. It is clear that, under decreasing  $\mu_2$ , a semi-stable limit cycle cannot appear in the domain  $D_2$ , since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation. By contradiction, we can prove that a semi-stable limit cycle cannot appear in the domain  $D_1$ .

Suppose it appears in this domain for some values of the parameters  $\mu_1^* > 0$  and  $\mu_2^* < 0$ . Return to system (2.2) and change the inputting order for the field rotation parameters. Input first the parameter  $\mu_2 < 0$ :

$$\dot{x} = P_n(x, y, \mu_2, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_2, 0, \dots, 0). \quad (2.5)$$

Fix it under  $\mu_2 = \mu_2^*$ . The vector field of (2.5) is rotated clockwise and the origin turns into an unstable focus. Inputting the parameter  $\mu_1 > 0$  into (2.5), we get again system (2.4) the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle,  $\Gamma_1$ , will appear from some separatrix cycle. The limit cycle  $\Gamma_1$  will contract, the outside spirals winding onto this cycle will untwist and the distance between their coils will increase under increasing  $\mu_1$  to the value  $\mu_1^*$ . It follows that there are no values of  $\mu_2^* < 0$  and  $\mu_1^* > 0$  for which a semi-stable limit cycle could appear in the domain  $D_1$ .

The second logical possibility can be excluded by controlling the stability character of the separatrix cycle [18]. Thus, only the first possibility is valid, i. e., system (2.4) has at most one limit cycle.

Let system (2.4) have the unique limit cycle  $\Gamma_1$ . Fix the parameters  $\mu_1 > 0$ ,  $\mu_2 < 0$  and input the third parameter,  $\mu_3 > 0$ , into this system supposing that  $\mu_3$  rotates its vector field counterclockwise:

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, \mu_3, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, \mu_3, 0, \dots, 0). \quad (2.6)$$

Here we can have two basic possibilities: 1) the limit cycle  $\Gamma_1$  disappears at the origin; 2) the second (unstable) limit cycle,  $\Gamma_2$ , appears from the origin and, under further increasing the parameter  $\mu_3$ , the cycle  $\Gamma_2$  joins with  $\Gamma_1$  forming a semi-stable limit cycle,  $\Gamma_{12}$ , which disappears in a “trajectory concentration” surrounding the origin. Besides, we can also suggest that: 3) in the domain  $D_2$  bounded by the origin and  $\Gamma_1$ , a semi-stable limit cycle,  $\Gamma_{23}$ , appears from a “trajectory concentration”, splits into an unstable cycle,  $\Gamma_2$ , and a stable cycle,  $\Gamma_3$ , and then the cycles  $\Gamma_1$ ,  $\Gamma_2$  disappear through a semi-stable limit cycle,  $\Gamma_{12}$ , and the cycle  $\Gamma_3$  disappears through an Andronov–Hopf bifurcation; 4) a semi-stable limit cycle,  $\Gamma_{34}$ , appears in the domain  $D_2$  bounded by the cycles  $\Gamma_1$ ,  $\Gamma_2$  and, for some set of values of the parameters,  $\mu_1^*$ ,  $\mu_2^*$ ,  $\mu_3^*$ , system (2.6) has at least four limit cycles.

Let us consider the last, fourth case. It is clear that a semi-stable limit cycle cannot appear either in the domain  $D_1$  bounded on the inside by the cycle  $\Gamma_1$

or in the domain  $D_3$  bounded by the origin and  $\Gamma_2$  because of the increasing distance between the spiral coils filling these domains under increasing the parameter  $\mu_3$ . To prove the impossibility of the appearance of a semi-stable limit cycle in the domain  $D_2$ , suppose the contrary, i. e., for some set of values of the parameters,  $\mu_1^* > 0$ ,  $\mu_2^* < 0$ , and  $\mu_3^* > 0$ , such a semi-stable cycle exists. Return to system (2.2) again and input first the parameters  $\mu_3 > 0$ ,  $\mu_1 > 0$ :

$$\dot{x} = P_n(x, y, \mu_1, \mu_3, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_3, 0, \dots, 0). \quad (2.7)$$

Fix these parameters under  $\mu_3 = \mu_3^*$ ,  $\mu_1 = \mu_1^*$  and input the parameter  $\mu_2 < 0$  into (2.7) getting again system (2.6). Since, by our assumption, this system has two limit cycles for  $\mu_2 > \mu_2^*$ , there exists some value of the parameter,  $\mu_2^{12}$  ( $\mu_2^* < \mu_2^{12} < 0$ ), for which a semi-stable limit cycle,  $\Gamma_{12}$ , appears in system (2.6) and then splits into a stable cycle,  $\Gamma_1$ , and an unstable cycle,  $\Gamma_2$ , under further decreasing  $\mu_2$ . The formed domain  $D_2$  bounded by the limit cycles  $\Gamma_1$ ,  $\Gamma_2$  and filled by the spirals will enlarge, since, by the properties of a field rotation parameter, the interior unstable limit cycle  $\Gamma_2$  will contract and the exterior stable limit cycle  $\Gamma_1$  will expand under decreasing  $\mu_2$ . The distance between the spirals of the domain  $D_2$  will naturally increase, what will prevent the appearance of a semi-stable limit cycle in this domain for  $\mu_2 < \mu_2^{12}$ .

Thus, there are no such values of the parameters,  $\mu_1^* > 0$ ,  $\mu_2^* < 0$ ,  $\mu_3^* > 0$ , for which system (2.6) would have an additional semi-stable limit cycle. Therefore, the fourth case cannot be realized. The third case is considered absolutely similarly. It follows from the first two cases that system (2.6) can have at most two limit cycles.

Suppose that system (2.6) has two limit cycles,  $\Gamma_1$  and  $\Gamma_2$ , fix the parameters  $\mu_1 > 0$ ,  $\mu_2 < 0$ ,  $\mu_3 > 0$  and input the fourth parameter,  $\mu_4 < 0$ , into this system supposing that  $\mu_4$  rotates its vector field clockwise:

$$\dot{x} = P_n(x, y, \mu_1, \dots, \mu_4, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \dots, \mu_4, 0, \dots, 0). \quad (2.8)$$

The most interesting logical possibility here is that when the third (stable) limit cycle,  $\Gamma_3$ , appears from the origin and then, under preservation of the cycles  $\Gamma_1$  and  $\Gamma_2$ , in the domain  $D_3$  bounded on the inside by the cycle  $\Gamma_3$  and on the outside by the cycle  $\Gamma_2$ , a semi-stable limit cycle,  $\Gamma_{45}$ , appears and then splits into a stable cycle,  $\Gamma_4$ , and an unstable cycle,  $\Gamma_5$ , i. e., when system (2.8) for some set of values of the parameters,  $\mu_1^*$ ,  $\mu_2^*$ ,  $\mu_3^*$ ,  $\mu_4^*$ , has at least five limit cycles. Logically, such a semi-stable limit cycle could also appear in the domain  $D_1$  bounded on the inside by the cycle  $\Gamma_1$ , since, under decreasing  $\mu_4$ , the spirals of the trajectories of (2.8) will twist and the distance between their coils will decrease. On the other hand, in the domain  $D_2$  bounded on the inside by the cycle  $\Gamma_2$  and on the outside by the cycle  $\Gamma_1$  and also in the domain  $D_4$  bounded by the origin and  $\Gamma_3$ , a semi-stable limit cycle cannot appear, since, under decreasing  $\mu_4$ , the spirals will untwist and the distance

between their coils will increase. To prove the impossibility of the appearance of a semi-stable limit cycle in the domains  $D_3$  and  $D_1$ , suppose the contrary, i. e., for some set of values of the parameters,  $\mu_1^* > 0$ ,  $\mu_2^* < 0$ ,  $\mu_3^* > 0$ , and  $\mu_4^* < 0$ , such a semi-stable cycle exists.

Return to system (2.2) again, input first the parameters  $\mu_4 < 0$ ,  $\mu_2 < 0$  and then the parameter  $\mu_1 > 0$ :

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, \mu_4, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, \mu_4, 0, \dots, 0). \quad (2.9)$$

Fix the parameters  $\mu_4, \mu_2$  under the values  $\mu_4^*, \mu_2^*$ , respectively. Under increasing  $\mu_1$ , a separatrix cycle is formed around the origin generating a stable limit cycle,  $\Gamma_1$ . Fix  $\mu_1$  under the value  $\mu_1^*$  and input the parameter  $\mu_3 > 0$  into (2.9) getting system (2.8).

Since, by our assumption, system (2.8) has three limit cycles for  $\mu_3 < \mu_3^*$ , there exists some value of the parameter  $\mu_3^{23}$  ( $0 < \mu_3^{23} < \mu_3^*$ ) for which a semi-stable limit cycle,  $\Gamma_{23}$ , appears in this system and then splits into an unstable cycle,  $\Gamma_2$ , and a stable cycle,  $\Gamma_3$ , under further increasing  $\mu_3$ . The formed domain  $D_3$  bounded by the limit cycles  $\Gamma_2, \Gamma_3$  and also the domain  $D_1$  bounded on the inside by the limit cycle  $\Gamma_1$  will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters  $\mu_1, \mu_2, \mu_3$ , and  $\mu_4$  are considered in a similar way. It follows that system (2.8) has at most three limit cycles. If we continue the procedure of successive inputting the field rotation parameters,  $\mu_5, \mu_6, \dots, \mu_k$ , into system (2.2), it is possible to conclude that system (2.1) can have at most  $k - 1$  limit cycles surrounding the origin. The theorem is proved.  $\square$

### 3 Limit cycles of the general Liénard polynomial system

By means of the same bifurcationally geometric approach, we will consider now the Liénard polynomial system (1.3). The study of singular points of system (1.3) will use two index theorems by H. Poincaré, see [1]. The definition of the Poincaré index is the following [1].

**Definition 3.1.** Let  $S$  be a simple closed curve in the phase plane not passing through a singular point of the system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (3.1)$$

where  $P(x, y)$  and  $Q(x, y)$  are continuous functions (for example, polynomials),

and  $M$  be some point on  $S$ . If the point  $M$  goes around the curve  $S$  in the positive direction (counterclockwise) one time, then the vector coinciding with the direction of a tangent to the trajectory passing through the point  $M$  is rotated through the angle  $2\pi j$  ( $j = 0, \pm 1, \pm 2, \dots$ ). The integer  $j$  is called the *Poincaré index* of the closed curve  $S$  relative to the vector field of system (3.1) and has the expression

$$j = \frac{1}{2\pi} \oint_S \frac{P dQ - Q dP}{P^2 + Q^2}. \quad (3.2)$$

According to this definition, the index of a node or a focus, or a center is equal to  $+1$  and the index of a saddle is  $-1$ .

**Theorem 3.1 (First Poincaré Index Theorem).** *If  $N$ ,  $N_f$ ,  $N_c$ , and  $C$  are respectively the number of nodes, foci, centers, and saddles in a finite part of the phase plane and  $N'$  and  $C'$  are the number of nodes and saddles at infinity, then it is valid the formula*

$$N + N_f + N_c + N' = C + C' + 1. \quad (3.3)$$

**Theorem 3.2 (Second Poincaré Index Theorem).** *If all singular points are simple, then along an isocline without multiple points lying in a Poincaré hemisphere which is obtained by a stereographic projection of the phase plane, the singular points are distributed so that a saddle is followed by a node or a focus, or a center and vice versa. If two points are separated by the equator of the Poincaré sphere, then a saddle will be followed by a saddle again and a node or a focus, or a center will be followed by a node or a focus, or a center.*

Consider system (1.3). Its finite singularities are determined by the algebraic system

$$x(1 + \beta_1 x + \dots + \beta_{2l} x^{2l}) = 0, \quad y = 0. \quad (3.4)$$

It always has an anti-saddle at the origin and, in general, can have at most  $2l+1$  finite singularities which lie on the  $x$ -axis and, according to Theorem 3.2, are distributed so that a saddle (or saddle-node) is followed by a node or a focus, or a center and vice versa. At infinity, system (1.3) has two singular points: a node at the “ends” of the  $x$ -axis and a saddle at the “ends” of the  $y$ -axis. For studying the infinite singularities, the methods applied in [1] for Rayleigh’s and van der Pol’s equations and also Erugin’s two-isocline method developed in [18] can be used (see [14], [15]).

Following [18], we will study limit cycle bifurcations of (1.3) by means of a canonical system containing field rotation parameters of (1.3) [1], [18].



**Theorem 3.3.** *The Liénard polynomial system (1.3) with limit cycles can be reduced to the canonical form*

$$\begin{aligned}\dot{x} &= y \equiv P(x, y), \\ \dot{y} &= -x(1 + \beta_1 x \pm x^2 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y(\alpha_0 + x + \alpha_2 x^2 + \dots + x^{2k-1} + \alpha_{2k} x^{2k}) \equiv Q(x, y),\end{aligned}\tag{3.5}$$

where  $\beta_1, \beta_3, \dots, \beta_{2l-1}$  are fixed and  $\alpha_0, \alpha_2, \dots, \alpha_{2k}$  are field rotation parameters of (3.5).

**Proof.** Let all the parameters  $\alpha_i$ ,  $i = 0, 1, \dots, 2k$ , vanish in system (3.5),

$$\dot{x} = y, \quad \dot{y} = -x(1 + \beta_1 x + \beta_2 x^2 + \dots + \beta_{2l} x^{2l}),\tag{3.6}$$

and consider the corresponding equation

$$\frac{dy}{dx} = \frac{-x(1 + \beta_1 x + \beta_2 x^2 + \dots + \beta_{2l} x^{2l})}{y} \equiv F(x, y).\tag{3.7}$$

Since  $F(x, -y) = -F(x, y)$ , the direction field of (3.7) (and the vector field of (3.6) as well) is symmetric with respect to the  $x$ -axis. It follows that for arbitrary values of the parameters  $\beta_j$ ,  $j = 1, 2, \dots, 2l$ , system (3.6) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, all the even parameters  $\beta_j$  of system (1.3) can be supposed to be equal, e. g., to  $\pm 1$ :  $\beta_2 = \beta_4 = \beta_6 = \dots = \pm 1$ .

Let now all the parameters  $\alpha_i$  with even indexes and  $\beta_j$  with odd indexes vanish in system (3.5),

$$\dot{x} = y, \quad \dot{y} = -x(1 \pm x^2 \pm \dots \pm x^{2l}) + y(\alpha_1 x + \alpha_3 x^3 + \dots + \alpha_{2k-1} x^{2k-1}),\tag{3.8}$$

and consider the corresponding equation

$$\begin{aligned}\frac{dy}{dx} &= \frac{-x(1 \pm x^2 \pm \dots \pm x^{2l}) + y(\alpha_1 x + \alpha_3 x^3 + \dots + \alpha_{2k-1} x^{2k-1})}{y} \\ &\equiv G(x, y).\end{aligned}\tag{3.9}$$

Since  $G(-x, y) = -G(x, y)$ , the direction field of (3.9) (and the vector field of (3.8) as well) is symmetric with respect to the  $y$ -axis. It follows that for arbitrary values of the parameters  $\alpha_1, \alpha_3, \dots, \alpha_{2k-1}$  system (3.6) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, all the odd parameters  $\alpha_i$  of system (1.3) can be supposed to be equal, e. g., to 1:  $\alpha_1 = \alpha_3 = \dots = \alpha_{2k-1} = 1$ .

Inputting the odd parameters  $\beta_1, \beta_3, \dots, \beta_{2l-1}$  into system (3.8),

$$\begin{aligned}\dot{x} &= y \equiv R(x, y), \\ \dot{y} &= -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y(x + x^3 + \dots + x^{2k-1}) \equiv S(x, y),\end{aligned}\tag{3.10}$$

and calculating the determinants

$$\begin{aligned}\Delta_{\beta_1} &= RS'_{\beta_1} - SR'_{\beta_1} = -x^2 y, \\ \Delta_{\beta_3} &= RS'_{\beta_3} - SR'_{\beta_3} = -x^4 y, \\ &\dots\dots\dots \\ \Delta_{\beta_{2l-1}} &= RS'_{\beta_{2l-1}} - SR'_{\beta_{2l-1}} = -x^{2l} y,\end{aligned}$$

we can see that the vector field of (3.10) is rotated symmetrically (in opposite directions) with respect to the  $x$ -axis and that the finite singularities (centers and saddles) of (3.10) moving along the  $x$ -axis (except the center at the origin) do not change their type or join in saddle-nodes. Therefore, we can fix the odd parameters  $\beta_1, \beta_3, \dots, \beta_{2l-1}$  in system (3.5), fixing the position of its finite singularities on the  $x$ -axis.

To prove that the even parameters  $\alpha_0, \alpha_2, \dots, \alpha_{2k}$  rotate the vector field of (3.5), let us calculate the following determinants:

$$\begin{aligned}\Delta_{\alpha_0} &= PQ'_{\alpha_0} - QP'_{\alpha_0} = y^2 \geq 0, \\ \Delta_{\alpha_2} &= PQ'_{\alpha_2} - QP'_{\alpha_2} = x^2 y^2 \geq 0, \\ &\dots\dots\dots \\ \Delta_{\alpha_{2k}} &= PQ'_{\alpha_{2k}} - QP'_{\alpha_{2k}} = x^{2k} y^2 \geq 0.\end{aligned}$$

By definition of a field rotation parameter [1], [18], for increasing each of the parameters  $\alpha_0, \alpha_2, \dots, \alpha_{2k}$ , under the fixed others, the vector field of system (3.5) is rotated in the positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (3.5) is rotated in the negative direction (clockwise).

Thus, for studying limit cycle bifurcations of (1.3), it is sufficient to consider the canonical system (3.5) containing only its even parameters  $\alpha_0, \alpha_2, \dots, \alpha_{2k}$  which rotate the vector field of (3.5) under the fixed others. The theorem is proved.  $\square$

By means of the canonical system (3.5), let us study global limit cycle bifurcations of (1.3) and prove the following theorem.

**Theorem 3.4.** *The general Liénard polynomial system (1.3) can have at most  $k + l$  limit cycles,  $k$  surrounding the origin and  $l$  surrounding one by one the other singularities of (1.3).*

**Proof.** According to Theorem 3.3, for the study of limit cycle bifurcations of system (1.3), it is sufficient to consider the canonical system (3.5) containing the field rotation parameters  $\alpha_0, \alpha_2, \dots, \alpha_{2k}$  of (1.3) under the fixed its parameters  $\beta_1, \beta_3, \dots, \beta_{2l-1}$ .

Let all these parameters vanish:

$$\dot{x} = y, \quad \dot{y} = -x(1 \pm x^2 \pm \dots \pm x^{2l}) + y(x + x^3 + \dots + x^{2k-1}). \quad (3.11)$$

System (3.11) is symmetric with respect to the  $y$ -axis and has centers as anti-saddles. Its center domains are bounded by either separatrix loops or digons of the saddles of (3.11) lying on the  $x$ -axis. If to input the parameters  $\beta_1, \beta_3, \dots, \beta_{2l-1}$  into (3.11) successively, we will get again system (3.10) the vector field of which is rotated symmetrically (in opposite directions) with respect to the  $x$ -axis. The finite singularities (centers and saddles) of (3.10) moving along the  $x$ -axis (except the center at the origin) do not change their type or join in saddle-nodes and the center domains will be bounded by separatrix loops of the saddles (or saddle-nodes) of (3.10) [1], [18].

Let us input successively the field rotation parameters  $\alpha_0, \alpha_2, \dots, \alpha_{2k}$  into system (3.10) beginning with the parameters at the highest degrees of  $x$  and alternating with their signs (see [14], [15]). So, begin with the parameter  $\alpha_{2k}$  and let, for definiteness,  $\alpha_{2k} > 0$ :

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y(x + x^3 + \dots + x^{2k-1} + \alpha_{2k} x^{2k}). \end{aligned} \quad (3.12)$$

In this case, the vector field of (3.12) is rotated in the positive direction (counterclockwise) turning the center at the origin into a nonrough (weak) unstable focus. All the other centers become rough unstable foci [1], [18].

Fix  $\alpha_{2k}$  and input the parameter  $\alpha_{2k-2} < 0$  into (3.12):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y(x + x^3 + \dots + \alpha_{2k-2} x^{2k-2} + x^{2k-1} + \alpha_{2k} x^{2k}). \end{aligned} \quad (3.13)$$

Then the vector field of (3.13) is rotated in the opposite direction (clockwise)

and the focus at the origin immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of  $x$  changes) generating a stable limit cycle. All the other foci will also generate stable limit cycles for some values of  $\alpha_{2k-2}$  after changing the character of their stability. Under further decreasing  $\alpha_{2k-2}$ , all the limit cycles will expand disappearing on separatrix cycles of (3.13) [1], [18].

Denote the limit cycle surrounding the origin by  $\Gamma_1$ , the domain outside the cycle by  $D_1$ , the domain inside the cycle by  $D_2$  and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a “trajectory concentration” surrounding this singular point. It is clear that, under decreasing the parameter  $\alpha_{2k-2}$ , a semi-stable limit cycle cannot appear in the domain  $D_2$ , since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation [14], [15].

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain  $D_1$ . Suppose it appears in this domain for some values of the parameters  $\alpha_{2k}^* > 0$  and  $\alpha_{2k-2}^* < 0$ . Return to system (3.10) and change the inputting order for the field rotation parameters. Input first the parameter  $\alpha_{2k-2} < 0$ :

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y(x + x^3 + \dots + \alpha_{2k-2} x^{2k-2} + x^{2k-1}).\end{aligned}\tag{3.14}$$

Fix it under  $\alpha_{2k-2} = \alpha_{2k-2}^*$ . The vector field of (3.14) is rotated clockwise and the origin turns into a nonrough stable focus. Inputting the parameter  $\alpha_{2k} > 0$  into (3.14), we get again system (3.13) the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle  $\Gamma_1$  will appear from a separatrix cycle for some value of  $\alpha_{2k}$ . This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing  $\alpha_{2k}$  to the value  $\alpha_{2k}^*$ . It follows that there are no values of  $\alpha_{2k-2}^* < 0$  and  $\alpha_{2k}^* > 0$  for which a semi-stable limit cycle could appear in the domain  $D_1$ .

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (3.13) for any values of the parameters  $\alpha_{2k-2}$  and  $\alpha_{2k}$  of different signs. Obviously, if these parameters have the same sign, system (3.13) has no limit cycles surrounding the origin at all. On the same reason, this system cannot have more than  $l$  limit cycles surrounding the other singularities (foci or nodes) of (3.13) one by one.

Let system (3.13) have the unique limit cycle  $\Gamma_1$  surrounding the origin and  $l$  limit cycles surrounding the other antisaddles of (3.13). Fix the parameters

$\alpha_{2k} > 0$ ,  $\alpha_{2k-2} < 0$  and input the third parameter,  $\alpha_{2k-4} > 0$ , into this system:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y(x + x^3 + \dots + \alpha_{2k-4} x^{2k-4} + \alpha_{2k-2} x^{2k-2} + x^{2k-1} + \alpha_{2k} x^{2k}).\end{aligned}\tag{3.15}$$

The vector field of (3.15) is rotated counterclockwise, the focus at the origin changes the character of its stability and the second (unstable) limit cycle,  $\Gamma_2$ , immediately appears from this point. The limit cycles surrounding the other singularities of (3.15) can only disappear in the corresponding foci (because of their roughness) under increasing the parameter  $\alpha_{2k-4}$ . Under further increasing  $\alpha_{2k-4}$ , the limit cycle  $\Gamma_2$  will join with  $\Gamma_1$  forming a semi-stable limit cycle,  $\Gamma_{12}$ , which will disappear in a “trajectory concentration” surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to  $\Gamma_{12}$ ? It is clear that such a limit cycle cannot appear either in the domain  $D_1$  bounded on the inside by the cycle  $\Gamma_1$  or in the domain  $D_3$  bounded by the origin and  $\Gamma_2$  because of the increasing distance between the spiral coils filling these domains under increasing the parameter  $\alpha_{2k-4}$  [14], [15].

To prove the impossibility of the appearance of a semi-stable limit cycle in the domain  $D_2$  bounded by the cycles  $\Gamma_1$  and  $\Gamma_2$  (before their joining), suppose the contrary, i. e., that for some set of values of the parameters,  $\alpha_{2k}^* > 0$ ,  $\alpha_{2k-2}^* < 0$ , and  $\alpha_{2k-4}^* > 0$ , such a semi-stable cycle exists. Return to system (3.10) again and input first the parameters  $\alpha_{2k-4} > 0$  and  $\alpha_{2k} > 0$ :

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y(x + x^3 + \dots + \alpha_{2k-4} x^{2k-4} + x^{2k-3} + \alpha_{2k} x^{2k}).\end{aligned}\tag{3.16}$$

Both parameters act in a similar way: they rotate the vector field of (3.16) counterclockwise turning the origin into a nonrough unstable focus.

Fix these parameters under  $\alpha_{2k-4} = \alpha_{2k-4}^*$ ,  $\alpha_{2k} = \alpha_{2k}^*$  and input the parameter  $\alpha_{2k-2} < 0$  into (3.16) getting again system (3.15). Since, by our assumption, this system has two limit cycles surrounding the origin for  $\alpha_{2k-2} > \alpha_{2k-2}^*$ , there exists some value of the parameter,  $\alpha_{2k-2}^{12}$  ( $\alpha_{2k-2}^* < \alpha_{2k-2}^{12} < 0$ ), for which a semi-stable limit cycle,  $\Gamma_{12}$ , appears in system (3.15) and then splits into a stable cycle,  $\Gamma_1$ , and an unstable cycle,  $\Gamma_2$ , under further decreasing  $\alpha_{2k-2}$ . The formed domain  $D_2$  bounded by the limit cycles  $\Gamma_1$ ,  $\Gamma_2$  and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle  $\Gamma_2$  will contract and the exterior stable limit cycle  $\Gamma_1$  will expand under decreasing  $\alpha_{2k-2}$ . The distance between the spirals of the domain  $D_2$  will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for  $\alpha_{2k-2} < \alpha_{2k-2}^{12}$  [14], [15].

Thus, there are no such values of the parameters,  $\alpha_{2k}^* > 0$ ,  $\alpha_{2k-2}^* < 0$ , and  $\alpha_{2k-4}^* > 0$ , for which system (3.15) would have an additional semi-stable limit cycle surrounding the origin. Obviously, there are no other values of the parameters  $\alpha_{2k}$ ,  $\alpha_{2k-2}$ , and  $\alpha_{2k-4}$  for which system (3.15) would have more than two limit cycles surrounding this singular point. On the same reason, additional semi-stable limit cycles cannot appear around the other singularities (foci or nodes) of (3.15). Therefore,  $2 + l$  is the maximum number of limit cycles in system (3.15).

Suppose that system (3.15) has two limit cycles,  $\Gamma_1$  and  $\Gamma_2$ , surrounding the origin and  $l$  limit cycles surrounding the other antisaddles of (3.15) (this is always possible if  $\alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} > 0$ ). Fix the parameters  $\alpha_{2k}$ ,  $\alpha_{2k-2}$ ,  $\alpha_{2k-4}$  and consider a more general system inputting the fourth parameter,  $\alpha_{2k-6} < 0$ , into (3.15):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y(x + x^3 + \dots + \alpha_{2k-6} x^{2k-6} + x^{2k-5} + \dots + \alpha_{2k} x^{2k}). \end{aligned} \quad (3.17)$$

For decreasing  $\alpha_{2k-6}$ , the vector field of (3.17) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating a third (stable) limit cycle,  $\Gamma_3$ . With further decreasing  $\alpha_{2k-6}$ ,  $\Gamma_3$  will join with  $\Gamma_2$  forming a semi-stable limit cycle,  $\Gamma_{23}$ , which will disappear in a “trajectory concentration” surrounding the origin; the cycle  $\Gamma_1$  will expand disappearing on a separatrix cycle of (3.17).

Let system (3.17) have three limit cycles surrounding the origin:  $\Gamma_1$ ,  $\Gamma_2$ ,  $\Gamma_3$ . Could an additional semi-stable limit cycle appear with decreasing  $\alpha_{2k-6}$  after splitting of which system (3.17) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain  $D_2$  bounded by the cycles  $\Gamma_1$  and  $\Gamma_2$  or in the domain  $D_4$  bounded by the origin and  $\Gamma_3$  because of the increasing distance between the spiral coils filling these domains after decreasing  $\alpha_{2k-6}$ . Consider two other domains:  $D_1$  bounded on the inside by the cycle  $\Gamma_1$  and  $D_3$  bounded by the cycles  $\Gamma_2$  and  $\Gamma_3$ . As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters  $\alpha_{2k}^* > 0$ ,  $\alpha_{2k-2}^* < 0$ ,  $\alpha_{2k-4}^* > 0$ , and  $\alpha_{2k-6}^* < 0$  such a semi-stable cycle exists. Return to system (3.10) again, input first the parameters  $\alpha_{2k-6} < 0$ ,  $\alpha_{2k-2} < 0$  and then the parameter  $\alpha_{2k} > 0$ :

$$\dot{x} = y,$$

$$\begin{aligned} \dot{y} = & -x(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \dots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ & + y(x + x^3 + \dots + \alpha_{2k-6} x^{2k-6} + \dots + \alpha_{2k-2} x^{2k-2} + x^{2k-3} + \alpha_{2k} x^{2k}). \end{aligned} \quad (3.18)$$

Fix the parameters  $\alpha_{2k-6}$ ,  $\alpha_{2k-2}$  under the values  $\alpha_{2k-6}^*$ ,  $\alpha_{2k-2}^*$ , respectively. With increasing  $\alpha_{2k}$ , a separatrix cycle formed around the origin will generate a stable limit cycle,  $\Gamma_1$ . Fix  $\alpha_{2k}$  under the value  $\alpha_{2k}^*$  and input the parameter  $\alpha_{2k-4} > 0$  into (3.18) getting system (3.17).

Since, by our assumption, (3.17) has three limit cycles for  $\alpha_{2k-4} < \alpha_{2k-4}^*$ , there exists some value of the parameter  $\alpha_{2k-4}^{23}$  ( $0 < \alpha_{2k-4}^{23} < \alpha_{2k-4}^*$ ) for which a semi-stable limit cycle,  $\Gamma_{23}$ , appears in this system and then splits into an unstable cycle,  $\Gamma_2$ , and a stable cycle,  $\Gamma_3$ , with further increasing  $\alpha_{2k-4}$ . The formed domain  $D_3$  bounded by the limit cycles  $\Gamma_2$ ,  $\Gamma_3$  and also the domain  $D_1$  bounded on the inside by the limit cycle  $\Gamma_1$  will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there [14], [15].

All other combinations of the parameters  $\alpha_{2k}$ ,  $\alpha_{2k-2}$ ,  $\alpha_{2k-4}$ , and  $\alpha_{2k-6}$  are considered in a similar way. It follows that system (3.17) can have at most  $3 + l$  limit cycles.

If we continue the procedure of successive inputting the even parameters,  $\alpha_{2k}, \dots, \alpha_2, \alpha_0$ , into system (3.10), it is possible first to obtain  $k$  limit cycles surrounding the origin ( $\alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} \gg -\alpha_{2k-6} \gg \alpha_{2k-8} \gg \dots$ ) and then to conclude that the canonical system (3.5) (i.e., the Liénard polynomial system (1.3) as well) can have at most  $k + l$  limit cycles,  $k$  surrounding the origin and  $l$  surrounding one by one the antisaddles (foci or nodes) of (3.5) (and (1.3) as well). The theorem is proved.  $\square$

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